Physics 618 2020

BCH, Some Remarks on Lie Groups, Heisenberg Extensions

April 17, 2020

Last time: Started to describe the Baker-Campbell-Hausdorff formula.

* $A, B \in M_n(k)$ $k = \mathbb{R}, \mathbb{C}$.

Find a formula for $C(A,B) = \log(e^A e^B)$

i.e. $$e^A e^B = e^C$$

Order $A \to tA$
$B \to tB$

$$C = A + B + \boxed{\text{∞ sum of nested commutators}}$$

Prelim: $\mathrm{Ad}(A) \in \mathrm{End}(M_n(k))$

$$\mathrm{Ad}(A)(B) := [A, B]$$

$$e^A B e^{-A} = e^{\mathrm{Ad}(A)} B$$

$A(t)$ matrix-valued function of t.

$$\frac{d}{dt} e^{A(t)} - \dot{A}(t) e^{A(t)} = \frac{1}{2}\left[A(t), \dot{A}(t)\right] + \cdots$$

$$\left(\frac{d}{dt} e^{A(t)}\right) e^{-A(t)} = \underbrace{f\big(\mathrm{Ad}(A(t))\big)}\, \dot{A}(t)$$

$$= \dot{A} + \frac{[A, \dot{A}]}{2} + \cdots$$

$$f(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^3}{3!} + \cdots$$

<u>pf</u>: $$B(s,t) := e^{sA(t)} \frac{d}{dt}\left(e^{-sA(t)}\right)$$

$B(0,t) = 0$ $\qquad$ $B(1,t) =$ what we want

$$\frac{\partial B}{\partial s} = \mathrm{Ad}\, A(t)\, B(s,t) - \dot{A}(t)$$

$$\left.\frac{\partial^j B}{\partial s^j}\right|_{s=0} = \cancel{(\mathrm{Ad}\, A(t))^j}\, \overset{\circ}{B} - \underline{(\mathrm{Ad}\, A(t))^{j-1} \dot{A}}$$

$\Longrightarrow$ formula from Taylor series @ $s=1$

$$\boxed{\frac{d}{dt} e^{A(t)} = \int_0^1 e^{sA(t)} \dot{A}(t) e^{(1-s)A(t)} ds}$$

$$\left(\frac{d}{dt} e^{A(t)}\right) e^{-A(t)} = \int_0^1 e^{sA(t)} \dot{A}(t) e^{-sA(t)} ds$$

$$= \int_0^1 e^{s\,Ad\,A(t)} ds\, \dot{A}(t)$$

$$= f\left(Ad(A(t))\right) \dot{A}(t)$$

$$\frac{d}{dt}\left[M_1(t) \cdots M_n(t)\right]$$

$$= \frac{dM_1(t)}{dt} M_2 \cdots M_n + M_1 \frac{dM_2}{dt} M_3 \cdots M_n$$

$$+ \cdots + M_1 \cdots M_{n-1} \frac{dM_n}{dt}$$

$$e^{A(t)} = \prod_{i=1}^{N} e^{\Delta s A(t)} \qquad \Delta s = \frac{1}{N}$$

$$\frac{d}{dt} e^{A(t)} = \sum_{s=1}^{N} \left(\prod_{i<s} e^{\Delta s A(t)}\right) \left(\underline{\frac{d}{dt} e^{\Delta s A(t)}}\right) \prod_{i>s} \cdots$$

$$\rightarrow \Delta s \dot{A}(t) + O(\Delta s)^2$$

Thm (BCH)

$$g(w) = \frac{\log w}{w-1} = \sum_{j=0}^{\infty} \frac{(1-w)^j}{j+1}$$

$$= 1 + \frac{1-w}{2} + \frac{(1-w)^2}{3} + \cdots$$

$$C = B + \int_0^1 g\left(e^{t\,\mathrm{Ad}A}\, e^{\mathrm{Ad}B}\right)(A)\,dt$$

$$g\,\underbrace{e^{\theta_1} e^{\theta_2}}_{}$$

power series $= 1 + \theta_1 + \theta_2 + \cdots$

Pf:

$$e^{C(t)} = e^{tA} e^{B}$$

$C(0) = B$ $\quad$ $C(1) =$ what we want

$$\boxed{e^{C(t)} \frac{d}{dt} e^{-C(t)} = -f\left(\mathrm{Ad}\,C(t)\right) \dot{C}(t)}$$

$$e^{C(t)} = e^{tA} e^{B}$$

$$e^{C(t)} \frac{d}{dt} e^{-C(t)} = e^{tA} e^{B} \frac{d}{dt}\left(e^{-B} e^{-tA}\right)$$

$$= e^{tA} \frac{d}{dt}\left(e^{-tA}\right) = -A$$

$$\Rightarrow \underbrace{f\left(\mathrm{Ad}\, C(t)\right)}_{\text{power series around } 1 \text{ in } \mathrm{Ad}\, C(t)} \dot{C}(t) = A$$

$$f(z) = \frac{e^{z}-1}{z} = 1 + \frac{z}{2!} + \cdots$$

$$\dot{C}(t) = \frac{1}{f\left(\mathrm{Ad}(C(t))\right)} A$$

$$g(w) = \frac{\log w}{w-1}$$

$$f(z)\, g(e^{z}) = \frac{e^{z}-1}{z} \, \frac{z}{e^{z}-1} = 1$$

$$\boxed{f(\theta)\, g(e^{\theta}) = 1} \qquad \theta = \mathrm{Ad}\, C(t)$$

$$\underline{\dot{C}(t)} = g\left(e^{\underline{\mathrm{Ad}\, C(t)}}\right) A$$

$$e^{C(t)} = e^{tA} e^{B} \Longrightarrow \boxed{e^{\mathrm{Ad}\, C(t)} = e^{t\mathrm{Ad}(A)} e^{\mathrm{Ad}(B)}}$$

$$C(t) = B + \int_0^1 g\left(e^{t\mathrm{Ad}(A)} e^{\mathrm{Ad}(B)}\right) dt \, (A)$$

$$= B + A + \frac{1}{2}[A,B]$$

$$+ \frac{1}{12}\left([A,[A,B]] + [B,[B,A]]\right) \leftarrow$$

$$+ \frac{1}{24}[A,[B,[A,B]]] + \cdots$$

Work to all orders in B and first order in A:

$$\boxed{C = B + \frac{\mathrm{Ad}(B)}{e^{\mathrm{Ad}(B)} - 1}(A) + \mathcal{O}(A^2)}$$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \qquad B_n = \text{Bernoulli \#'s}$$

$$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

$A \in M_n(k)$

$$A \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\mathrm{Ad}(A) : M_n(k) \longrightarrow M_n(k)$$

is diagonalizable, e.v.'s $\boxed{\lambda_i - \lambda_j}$ $1 \leq i, j \leq n$.

So if $\mathrm{Ad}(B)$ has $\lambda_i - \lambda_j = \underline{2\pi\sqrt{-1}\,n}$ $n \in \mathbb{Z}$ if $n \neq 0$

then $\dfrac{\mathrm{Ad}(B)}{e^{\mathrm{Ad}(B)} - 1}$ will not converge..

BCH $\quad C = A + B + \left[\text{Sum of Nested Commutators}\right]$

$\mathfrak{g} \subset M_n(k)$

linear subspace.

$e^A e^B = e^C \qquad C \in \mathfrak{g}$

(if series converges)

Closure under commutator.

Abstractly Lie Algebra / k
is a vectorspace $\mathfrak{g}$ with a
bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

antisymmetric + satisfies Jacobi

$$[x,[y,z]] + \text{cyclic} = 0$$

But for $M_n(k)$ we can take
matrix commutator and if we have
a linear subspace $\mathfrak{g} \subset M_n(k)$
<u>closed</u> <u>under</u> <u>matrix</u> <u>commutator</u>

Then we have a Lie algebra

$$e^C = e^A e^B$$

exponentiating matrices from $\mathfrak{g}$
(locally) forms a Lie group.

$$\underline{e^A}\;\underline{e^B} = e^{\underbrace{A+B}_{\mathfrak{g}} + \underbrace{\text{sum com's}}_{\mathfrak{g}}}$$

$A, B \in \mathfrak{g}$

Surjectivity, radius of convergence.

Compact Lie groups: exp map is <u>onto</u>

So

$$\{ \underline{\underline{e}}^A \mid A \in \mathfrak{g} \} = G.$$

(a.) $\mathfrak{g} = gl(n,k) = M_n(k)$

closed under matrix commutator

(b.) $\mathfrak{g} = su(n) = A_{n-1}$

$= \{ n \times n$ traceless <u>antiHermitian</u> matrices. $\}$

$$(e^A)^+ = e^{A^+} = e^{-A} \quad \text{unitary}$$

$$\{ e^A \mid A \in su(n) \} = SU(n)$$

(c.) $\mathfrak{g} = so(m)$ $m \times m$ real antisymmetric closed under matrix commutator

$$\{e^A\} = SO(m)$$

$$so(2n+1) = B_n$$

$$so(2n) = D_n$$

(d) $\exp\{a \in M_{2n}(\mathbb{C}) \mid (Ja)^{tr} = Ja\}$

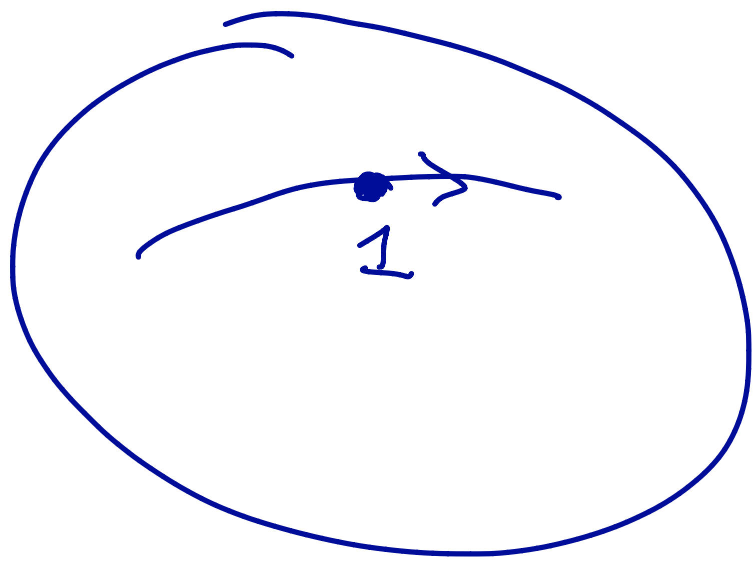
closed

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

$sp(n)$ or $usp(2n)$

or C_n

$$Sp(n) = USp(2n)$$
$$= Sp(2n, \mathbb{C}) \cap U(2n).$$

$G =$ 

$$\boxed{T_1 G = \mathfrak{g}}$$

$$g_1 = e^{t_1 A_1} \qquad g_2 = e^{t_2 A_2}$$

$$\underbrace{g_1 g_2 g_1^{-1} g_2^{-1}} = 1 + t_1 t_2 \underbrace{[A_1, A_2]} + \cdots$$

Representations :

Def: A rep. of a Lie algebra (V, ρ)

$$\rho : \underbrace{\mathfrak{g}} \longrightarrow \mathrm{End}(V) = \mathrm{Hom}(V, V)$$

$$= \{ \text{linear trans } V \to V \}$$

Lie algebra homom.

$$[\rho(x), \rho(y)] = \rho\Big(\underbrace{[x, y]}_{\substack{\text{abst. comm.} \\ \text{in } \mathfrak{g}}}\Big)$$

A rep. of a Lie group (V, ρ)

$$\rho : G \longrightarrow GL(V)$$

group homomorphism

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$$

BCH: If $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homom. then

(*) $$\boxed{\rho(e^A) := e^{\dot{\rho}(A)}}$$ is a Lie group homomorphism.

Given a Lie algebra there is <u>always</u> a canonical representation

$$V = \mathfrak{g} \ni x$$

$$\dot{\rho}(x) \in \mathrm{End}(V) = \mathrm{End}(\mathfrak{g})$$

$$\dot{\rho}(x)(y) = [x, y] \in \mathfrak{g}.$$

Called the <u>adjoint representation</u>

$$\dot{\rho}(x) = \mathrm{Ad}(x)$$

$$e^A B e^{-A} = e^{\mathrm{Ad}(A)} B$$

$$\boxed{\mathrm{Ad}(e^A)(B) = e^{\mathrm{Ad}(A)} B}$$ special case of (*)

If $\mathrm{Ad}(A)$ $\mathrm{Ad}(B)$ are nilpotent the series will terminate

$$\exp\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & x' & y' \\ 0 & 0 & z' \\ 0 & 0 & 0 \end{pmatrix}$$

Heisenberg Groups:

Basic motivating example

Q.M. of particle on a line $\mathbb{R}$

$$[\hat{q}, \hat{p}] = i\hbar$$

Operator algebra — we ask about rep's.

One standard rep $\mathcal{H} = L^2(\mathbb{R})$

$$(\hat{q}\cdot\psi)(q) = q\,\psi(q) \qquad q \in \mathbb{R}$$

$$(\hat{p}\cdot\psi)(q) = -i\hbar \frac{d}{dq}\psi(q)$$

$$U(\alpha) = \exp(i\alpha\hat{p})$$ $\alpha \in \mathbb{C}$ or $\mathbb{R}$

$$V(\beta) = \exp(i\beta\hat{q})$$

$$U(\alpha_1)U(\alpha_2) = U(\alpha_1+\alpha_2)$$

$$\boxed{U(\alpha)V(\beta) = e^{i\hbar\alpha\beta}\, V(\beta)U(\alpha)}$$

3 - proofs:

① BCH $[\hat{p},\hat{q}] =$ is central

$\therefore$ all higher terms vanish.

② $(U(\alpha)V(\beta)\cdot\psi)(q) = (V(\beta)\psi)(q+\hbar\alpha)$

$$= e^{i\beta(q+\hbar\alpha)}\,\psi(q+\hbar\alpha)$$

$$(V(\beta)U(\alpha)\cdot\psi)(q) = e^{i\beta q}\,(U(\alpha)\psi)(q)$$

$$= e^{i\beta q}\,\psi(q+\hbar\alpha)$$

$e^{i\alpha\beta\hbar}$

$$U(\alpha)\, e^{i\beta\hat{q}}\, U(\alpha)^{-1}$$

$$= e^{i\beta\, U(\alpha)\hat{q}U(\alpha)^{-1}}$$

$$= e^{i\beta\left(e^{i\alpha \mathrm{Ad}(\hat{p})}\hat{q}\right)}$$

$$= e^{i\beta\left(\hat{q} + \hbar\alpha \mathbb{1}\right)}$$

$$= e^{i\hbar\alpha\beta}\, V(\beta)$$

Consider group $\mathcal{O}$ of operators on $\mathcal{H}$ generated by $U(\alpha), V(\beta)$ $\alpha, \beta \in \mathbb{R}$.

$$U(\alpha_1)V(\beta_1)U(\alpha_2)V(\beta_2)\cdots$$

$$= e^{i\hbar(\text{quadratic in } \alpha_i, \beta_j)}$$

$$U(\alpha_1 + \cdots + \alpha_n)\, V(\beta_1 + \cdots + \beta_n)$$

$$1 \to U(1) \to \underline{\vartheta} \underset{\xleftarrow{\;s\;}}{\xrightarrow{\;\pi\;}} \mathbb{R}\oplus\mathbb{R} \to 1$$

as an abstract group it is isomophic to

$$\big(\underset{\substack{\cap\\ U(1)}}{z_1}, (\alpha_1,\beta_1)\big)\cdot\big(z_2,(\alpha_2,\beta_2)\big)$$

$$= \Big(z_1 z_2 \underbrace{e^{\frac{i}{2}\hbar(\alpha_1\beta_2-\alpha_2\beta_1)}}_{\boxed{\text{made a choice of cocycle.}}},\ (\alpha_1+\alpha_2,\ \beta_1+\beta_2)\Big)$$

$$\vartheta \cong \mathrm{Heis}(\mathbb{R}\oplus\mathbb{R})$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = v_1^{tr} J v_2$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

$$\omega(v_1, v_2) = v_1^{tr} J v_2 = -\omega(v_2, v_1)$$

ω is a symplectic form on $\mathbb{R}\oplus\mathbb{R}$

Invt under the Symplectic group

$Sp(2,\mathbb{R}) \ni A$

i.e. $\omega(Av_1, Av_2) = \omega(v_1, v_2)$

$s(\alpha,\beta) = U(\alpha)V(\beta) \longleftarrow$

$\hat{S}(\alpha,\beta) = \exp(i(\alpha\hat{p} + \beta\hat{q}))$

gives the above cocycle.

$1 \to \underline{U(1)} \to \mathrm{Heis}(\mathbb{R}^n\oplus\mathbb{R}^n) \to \underline{\mathbb{R}^n\oplus\mathbb{R}^n} \to 1$

$\underbrace{U(1) \cong Z(\mathrm{Heis}(\mathbb{R}^n\oplus\mathbb{R}^n))}$

$(z_1, \vec{v}_1)(z_2, \vec{v}_2) :=$

$$:= \left(z_1 z_2 \underline{\underline{e^{i\frac{\hbar}{2}\omega(v_1,v_2)}}}, v_1+v_2\right)$$

$\omega(v_1,v_2) = v_1^{tr} J_n v_2 = -\omega(v_2, v_1)$

Nondeg: $v_1 \neq 0 \ \exists v_2 \ \omega(v_1,v_2) \neq 0$.

If $[\hat{q}^i, \hat{p}_j] = i\hbar \delta^i_j$

$U(\vec{\alpha})$, $V(\vec{\beta})$ generate a group isomorphic to $\mathrm{Heis}(\mathbb{R}^n \oplus \mathbb{R}^n)$

$$s(\vec{\alpha}, \vec{\beta}) = \exp\left[i(\alpha^i \hat{p}_i + \beta_i \hat{q}^i)\right]$$

$U(\vec{\alpha})V(\vec{\beta})$ or $V(\vec{\beta})U(\vec{\alpha})$ give cocycles differing by coboundary.

General Heisenberg Groups

Consider central extensions of an Abelian group G by an Abelian group A.

$$\boxed{1 \to A \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1}$$

c.e. and G Abelian

it can happen that $\widetilde{G}$ is nonabelian

Examples D_4, Q as exts of $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\sigma_1 \quad \sigma_2}$ by $\underbrace{\mathbb{Z}_2}_{\sigma_3}$, $\mathrm{Heis}(\mathbb{R}^n \oplus \mathbb{R}^n)$

Assume we know cocycle $f(g_1, g_2)$ defining $\tilde{G}$ and

Compute group commutator

$$\Longrightarrow \left[\underbrace{(a_1, g_1)}_{\in \tilde{G}}, \underbrace{(a_2, g_2)}_{\in \tilde{G}} \right]$$

group commutator in $\tilde{G}$

you compute!

$$= \left(\frac{f(g_1, g_2)}{f(g_2, g_1)}, 1 \right)$$

because G Abelian

$$k(g_1, g_2) = \frac{f(g_1, g_2)}{f(g_2, g_1)}.$$

$$\boxed{k : G \times G \longrightarrow A \quad \text{"Commutator function"}}$$

$f : G \times G \longrightarrow A$

Nice Properties Of k

1.) k is gauge-invariant.

$$k(g_1,g_2) = \frac{f(g_1,g_2)}{f(g_2,g_1)} = \frac{\tilde{f}(g_1,g_2)}{\tilde{f}(g_2,g_1)}$$

if $\tilde{f} \sim f$ by coboundary.

2.) $k(g, 1_G) = k(1_G, g) = 1_A \in A$ ✓

3.) Extension $\tilde{G}$ is Abelian $\Leftrightarrow$ iff $k = 1$ (i.e. $\exists$ symmetric cocycle $f(g_1,g_2) = f(g_2,g_1)$)

4.) k skew $k(g_1,g_2) = k(g_2,g_1)^{-1}$

5.) k alternating $k(g,g) = 1$.

6.) k is bimultiplicative.

$$k(g_1 g_2, g_3) = k(g_1, g_3)\, k(g_2, g_3)$$

$$k(g_1, g_2 g_3) = k(g_1, g_2)\, k(g_1, g_3)$$

last property is not manifest, but it follows from the cocycle identity for f.

<u>Def</u>: A general <u>Heisenberg extension</u> is an extension <u>of the above form</u> where k is <u>nondegenerate</u>, (G, A Abelian) i.e. if $g_1 \neq 1$ $\exists g_2$ $k(g_1, g_2) \neq 1$.

In this case

$$Z(\tilde{G}) \cong A$$

So $\tilde{G}$ is maximally nonabelian.

In general, k nondeg: Heisenberg $k=1$ $\tilde{G} = A \times G$

$$A \subset Z(\tilde{G}) \subset \tilde{G}$$

For $\mathrm{Heis}(\mathbb{R}^n \oplus \mathbb{R}^n)$

$$\boxed{k(v_1, v_2) = \exp\left(\frac{i}{\hbar}\,\omega(v_1, v_2)\right).}$$

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$$e^{C(t)} = e^{tA} e^{B}$$

$$\Longrightarrow \quad e^{\mathrm{Ad}\, C(t)} = e^{t\mathrm{Ad}(A)} e^{\mathrm{Ad}(B)}$$

group homom. property of adjoint rep.

$$e^{A} e^{B} \qquad e^{(\underline{X+t\alpha})} e^{(\underline{Y+t\beta})}$$

expand around

$$= \boxed{e^{X} e^{Y} + \text{Series in } t}$$

$$= e^{C(X,Y) + \text{series in } t}$$

$$X + t\alpha + \int_0^1 ds\, g\left(e^{s(\mathrm{Ad}X + t\mathrm{Ad}\alpha)} e^{\mathrm{Ad}Y + t\mathrm{Ad}\beta}\right)$$

$$(\cancel{\alpha}\ Y + t\beta)$$

$$e^{-t\alpha} e^{X+t\alpha} e^{Y+t\beta} e^{-t\beta}$$

$$\underline{\underline{e^{X + O(t^2)}}} \quad \underline{\underline{e^{Y + O(t^2)}}}$$